The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 ***Solution 2.0***

Date: September 26, 2023 Course: EE 313 Evans

Name: ____
Last, First

- This in-person exam is scheduled to last 75 minutes.
- Open books, open notes, and open class materials, including homework assignments and solution sets and previous midterm exams and solutions.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please mute all computer systems.
- Please turn off all phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation**.**

Problem	*Point Value*	*Your score*	*Topic*
1	25		Sampling Sinusoids
2	25		Fourier Series
3	26		Sampling
4	24		Spectrograms
Total	100		

Problem 1.1 *Sampling Sinusoids*. 25 points.

Fall 2017, 2018 & 2021 Midterm 1.1

SPFirst Sec. 2-1 & 2-3

SPFirst Sec. 4-1 & 4-2

Homework Prob. 3.2

Lecture slides 5-5 & 5-6

Lecture slides 6-4 to 6-7

Lecture 5 Notes Sep. 21st

Tune-Up Tuesday #1

Consider the sinusoidal signal $x(t) = \cos(2\,\pi\,f_0\,t)$ for continuous-time frequency f_0 in Hz.

We are able to observe $x(t)$ for all time, i.e. for $-\infty < t < \infty$.

We sample $x(t)$ at a sampling rate of f_s in Hz to produce a discrete-time signal $x[n]$.

(a) Derive the formula for $x[n]$ by sampling $x(t)$ at a sampling rate of f_s in Hz. *6 points.*

$$\boldsymbol{x[n] = x(t)|_{t=nT_s} = \cos\left(2\pi f_0 (nT_s)\right) = \cos(2\pi f_0 T_s n) = \cos\left(2\pi \frac{f_0}{f_s} n\right)}$$

$$\boldsymbol{x[n] = \cos(\hat{\omega}_0 n)}$$

(b) Based on your answer in part (a), give a formula for the discrete-time frequency $\hat{\omega}_0$ of $x[n]$ in terms of the continuous-time frequency f_0 and sampling rate f_s. Units of $\hat{\omega}_0$ are in rad/sample. *6 points*.

$$\boldsymbol{\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}}$$

(c) We choose the sampling rate f_s to satisfy the Sampling Theorem, i.e. $f_s > 2\,f_0$.

i. Give the range of continuous-time frequencies in Hz that can be correctly captured by sampling. Be sure to include negative, positive and zero frequencies. Justify your reasoning. *6 points.*

Sampling Theorem says if $x(t)$ is sampled using a sampling rate $f_s > 2\,f_0$ where f_0 is the maximum frequency in $x(t)$, then we can reconstruct $x(t)$ from its samples.

With $f_s > 2\,f_0$, we can divide both sides by 2 and obtain

$$\boldsymbol{f_0 < \frac{1}{2}\,f_s}$$

For every positive frequency component f_0 in $x(t) = \cos(2\,\pi\,f_0\,t)$, we have a $-f_0$ component because we can write cosine as a sum of two phasors (complex sinusoids):

$$\boldsymbol{\cos(2\,\pi\,f_0\,t) = \frac{1}{2}\,e^{-j\,2\,\pi\,f_0\,t} + \frac{1}{2}\,e^{j\,2\,\pi\,f_0\,t}}$$

The range of continuous-time frequencies in Hz that can be correctly captured by sampling is

$$\boldsymbol{-\frac{1}{2}\,f_s < f < \frac{1}{2}\,f_s}$$

ii. Give the range of discrete-time frequencies that are represented when the Sampling Theorem is satisfied, i.e. $f_s > 2\,f_0$. Be sure to include negative, positive and zero frequencies. Justify your reasoning. *7 points*

Using the result from (b), $\hat{\omega}_0 = 2\pi\frac{f_0}{f_s}$, we let $f_0 \to -\frac{1}{2}\,f_s$ to obtain $\hat{\omega}_0 = -\pi$ and $f_0 \to \frac{1}{2}\,f_s$ to obtain $\hat{\omega}_0 = \pi$. So, the range is $-\pi < \hat{\omega}_0 < \pi$.

Note: In order for the discrete-time frequency domain to be periodic with period 2π, either $-\pi$ or π would need to be included in the interval (but not both). Several students had this more correct answer, but full marks were also given for $-\pi < \hat{\omega}_0 < \pi$.

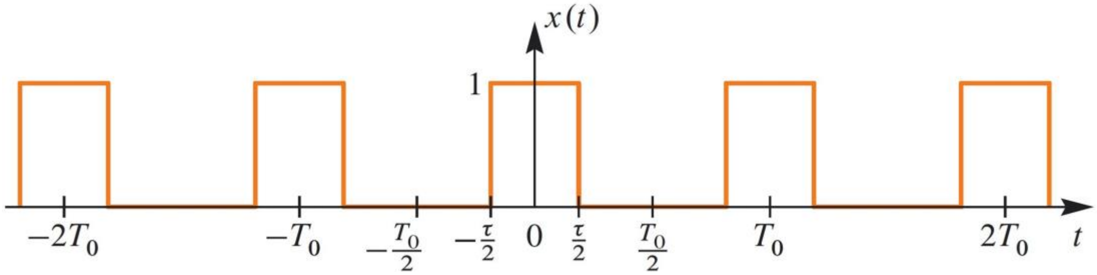
Fall 2017 & 2018 Midterm 1.3 | SPFirst Sec. 3-3 to 3-6
Fourier Series for a Square Wave in SPFirst Sec. 3-6.1
Lecture slides 3-7 to 3-14 | Homework Prob. 3.1

Problem 1.2 *Fourier Series.* 25 points.

Compute the Fourier series for a periodic pulse wave

$$x(t) = \begin{cases} 1 & \text{for } 0 \le |t| < \frac{\tau}{2} \\ 0 & \text{for } \frac{\tau}{2} \le |t| < \frac{T_0}{2} \end{cases}$$

The fraction of the time the pulse is "on" (i.e. has value 1) in each fundamental period T_0 is $\frac{\tau}{T_0}$.



(a) Compute Fourier series coefficients a_k of $x(t)$ to represent $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j 2\pi k f_0 t}$

i. Compute a_0. *6 points.* $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt$

a_0 is the average value of $x(t)$ which is the area of $x(t)$ over one fundamental period (which is the base times height of the rectangle = τ) divided by T_0 :

$$\boldsymbol{a_0 = \frac{\tau}{T_0}}$$

ii. Compute a_k for $k \ne 0$. *12 points.* $a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi k f_0 t}\, dt$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{-\tau} (0)\, e^{-j2\pi k f_0 t}\, dt + \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} (1)\, e^{-j2\pi k f_0 t}\, dt + \frac{1}{T_0}\int_{\tau/2}^{T_0/2} (0)\, e^{-j2\pi k f_0 t}\, dt$$

sinc function

$$a_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} e^{-j2\pi k f_0 t}\, dt = -\frac{e^{-j2\pi k f_0 t}}{j2\pi k \cancel{f_0 T_0}}\Bigg]_{-\tau/2}^{\tau/2} = \frac{e^{-j2\pi k f_0(\tau/2)} - e^{-j2\pi k f_0(-\tau/2)}}{j2\pi k} = \boxed{\frac{\sin(\pi k\, \tau/T_0)}{\pi k}}$$

(b) Compute Fourier series coefficients b_k of $y(t) = 2\, x(t) - 2\frac{\tau}{T_0}$ to represent $y(t) = \sum_{k=-\infty} b_k\, e^{j2\pi k f_0 t}$

i. Compute b_0. *3 points.* $b_0 = \frac{1}{T_0}\int_0^{T_0} y(t)\, dt$

$$\boxed{\boldsymbol{b_0 = 0}}$$

$$b_0 = \frac{1}{T_0}\int_0^{T_0}\left(2\, x(t) - 2\frac{\tau}{T_0}\right) dt = 2\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, dt\right) - \left(2\frac{\tau}{T_0}\right)\left(\frac{1}{T_0}\int_0^{T_0} dt\right) = 2\, a_0 - 2\frac{\tau}{T_0} = 0$$

ii. Compute b_k for $k \ne 0$. *4 points.* $b_k = \frac{1}{T_0}\int_0^{T_0} y(t)\, e^{-j2\pi k f_0 t}\, dt$

$$\boxed{\boldsymbol{b_k = 2\, a_k}}$$

$$b_k = \frac{1}{T_0}\int_0^{T_0}\left(2\, x(t) - 2\frac{\tau}{T_0}\right) e^{-j2\pi k f_0 t}\, dt = 2\underbrace{\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt\right)}_{a_k} - 2\frac{\tau}{T_0}\underbrace{\left(\frac{1}{T_0}\int_0^{T_0} e^{-j2\pi k f_0 t}\, dt\right)}_{0}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{-j2\pi k f_0 t}\, dt = -\frac{e^{-j2\pi k f_0 t}}{j2\pi k \cancel{f_0 T_0}}\Bigg]_0^{T_0} = \frac{e^{-j2\pi k \cancel{f_0 T_0}} - e^{-j2\pi k f_0 (0)}}{j2\pi k} = \frac{e^{-j2\pi k} - 1}{j2\pi k} = 0$$

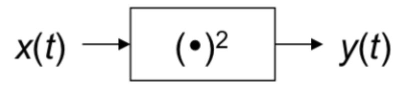
Fall 2018 Final Exam Prob 8(c) | Homework Prob. 2.1 & 3.1

SPFirst Sec. 1-2 & 1-3 | Lecture Slide 1-20

SPFirst Sec. 2-2 & 2-4 | Lecture Slides 2-5 & 2-6

Problem 1.3. *Sampling*. 26 points**.**

(a) Let $x(t) = \cos(2\pi f_0 t)$ be a continuous-time signal for $-\infty < t < \infty$.



i. From the block diagram below, derive a formula for $y(t)$ and write it as a sum of cosines. *6 points*.

$x(t)$ has frequencies $-f_0$ and $+f_0$ because $x(t) = \cos(2\pi f_0 t) = \frac{1}{2}e^{-j2\pi f_0 t} + \frac{1}{2}e^{j2\pi f_0 t}$

$$y(t) = x^2(t) = \cos^2(2\pi f_0 t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi(2f_0)t)$$

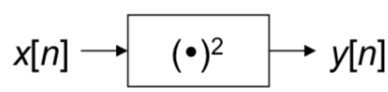
ii. Let $f_0 = 3000\ Hz$. What negative, zero, and positive frequencies are present in $y(t)$? *6 points*

$y(t)$ has continuous-time frequencies of $-2\,f_0$, 0, and $+2\,f_0$. Here, $f_{\max} = 2\,f_0$.

For $f_0 = 3000$ Hz, $y(t)$ has frequencies -6000 Hz, 0 Hz, and 6000 Hz.

Here, $f_{\max} = 6000$ Hz.

(b) Let $x(t) = \cos(2\pi f_0 t)$ be a continuous-time signal for $-\infty < t < \infty$.



i. From the block diagram below, derive a formula for $y[n]$and write it as a sum of cosines. *6 points*

Due to sampling at sampling rate f_s where $f_s = \frac{1}{T_s}$ and T_s is the sampling time,

$$x[n] = x(t)]_{t = n\,T_s} = \cos(2\pi f_0 (n\,T_s)) = \cos\left(2\pi f_0 \left(\frac{n}{f_s}\right)\right) = \cos\left(\left(2\pi \frac{f_0}{f_s}\right) n\right)$$

The discrete-time frequency is $\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}$ in rad/sample.

$x[n]$ has frequencies $-\hat{\omega}_0$ and $+\hat{\omega}_0$ because $x[n] = \cos(\hat{\omega}_0 n) = \frac{1}{2}e^{-j\hat{\omega}_0 n} + \frac{1}{2}e^{j\hat{\omega}_0 n}$

$$y[n] = x^2[n] = \cos^2(\hat{\omega}_0 n) = \frac{1}{2} + \frac{1}{2}\cos(2\hat{\omega}_0 n)$$

ii. Let $f_0 = 3000$ Hz and $f_s = 8000$ Hz. What negative, zero and positive discrete-time frequencies are present in $y[n]$ between –π rad/sample and π rad/sample? What are their corresponding continuous-time frequencies? *8 points*.

Aliasing will occur because the maximum frequency in $y(t)$ is 6000 Hz and the sampling rate f_s does not satisfy $f_s > 2\,f_{\max}$.

For $x[n] = \cos(\hat{\omega}_0 n)$, $\hat{\omega}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{3000\text{ Hz}}{8000\text{ Hz}} = 2\pi \frac{3}{8}$ rad/sample

SPFirst Sec. 4-1

For $y[n] = \frac{1}{2} + \frac{1}{2}\cos(2\hat{\omega}_0 n)$ and

$$\cos(2\hat{\omega}_0 n) = \cos\left(2\left(2\pi\frac{3}{8}\right)n\right) = \cos\left(2\pi\frac{6}{8}n\right) = \cos\left(2\pi\frac{6}{8}n - 2\pi n\right) = \cos\left(-2\pi\frac{2}{8}n\right)$$

Due to aliasing, $y[n]$ has discrete-time frequencies of $-2\pi\frac{2}{8}$, 0, and $2\pi\frac{2}{8}$ rad/sample

With $\hat{\omega}_1 = 2\pi f_1 / f_s$ and hence $f_1 = (\hat{\omega}_1/(2\pi))\,f_s$, the corresponding continuous-time frequencies are -2000 Hz, 0 Hz, and 2000 Hz.

Problem 1.4. *Spectrograms*. 24 points.

Below are spectrograms (labeled with numbers 1-6) for six signals over the time interval $0 \le t \le 2s$.

The vertical frequency axis for each spectrogram shows non-negative frequencies in units of Hz.

For each signal (a)-(f), identify the corresponding spectrogram and explain your reasoning.

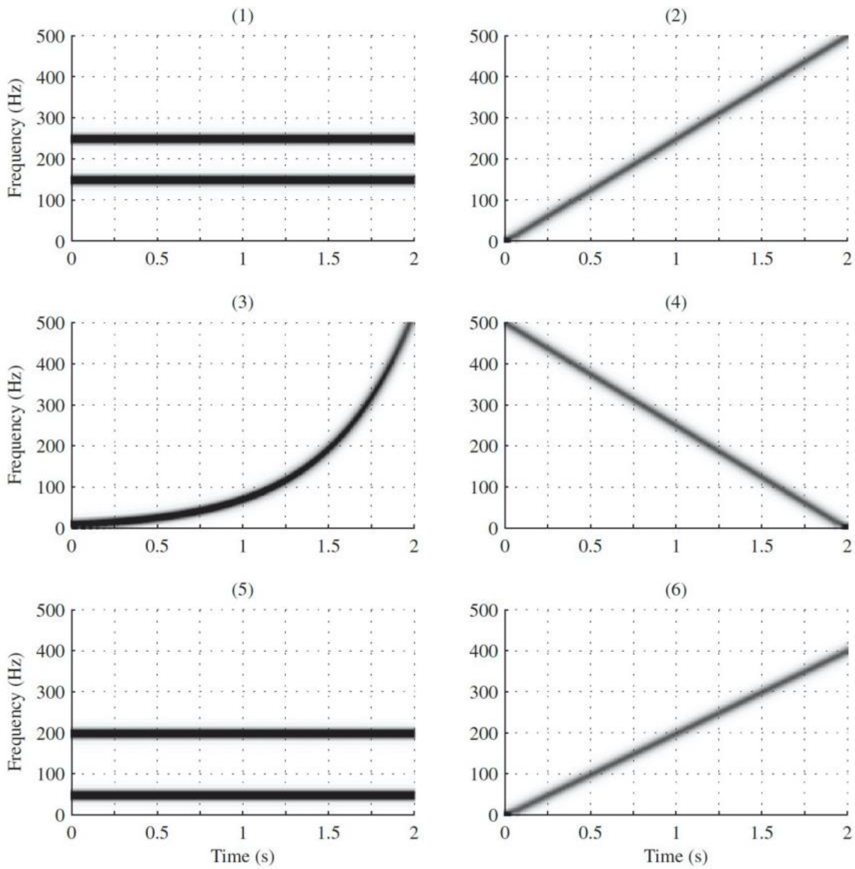
Mini-Project #1

Fall 2017 Midterm 1.4

SPFirst Sec. 3-7 & 3-8

Lecture Slides 4-4 & 4-12

Homework Prob. 2.3 & 2.4



The spectrogram plots the magnitude of the frequency components vs. time in a signal. Phase is ignored.

The frequency components are related to the instantaneous frequencies in the signal, i.e., the frequency at a particular time. For $x(t) = \cos(\theta(t))$, the instantaneous frequency is $\frac{d\theta(t)}{dt}$ in rad/s.

The spectrograms on the right only show the positive frequency content; the negative frequencies would be a mirror image of positive frequencies.

(a) $x(t) = \cos(-250\pi\, t^2)$. **Instantaneous frequency is $\frac{d\theta(t)}{dt} = \frac{d}{dt}(-250\pi\, t^2) = -500\pi t$ rad/s which is a line from 0 Hz to -500 Hz for $0 \le t \le 2s$. Every negative frequency has a positive frequency, which would be a line from 0 Hz to 500 Hz. This is spectrogram (2).**

(b) $x(t) = \cos\left(100\pi\, t - \frac{\pi}{4}\right) + \cos(400\pi\, t)$. **Sum of 50 and 200 Hz components. Spectrogram (5).**

(c) $x(t) = \cos(1000\pi\, t - 250\pi\, t^2)$. **$\frac{d\theta(t)}{dt} = \frac{d}{dt}(1000\pi\, t - 250\pi\, t^2) = 1000\pi - 500\pi t$ rad/s which is a line from 500 Hz to 0 Hz for $0 \le t \le 2s$. This is spectrogram (4).**

Tune-Up Tuesday #3

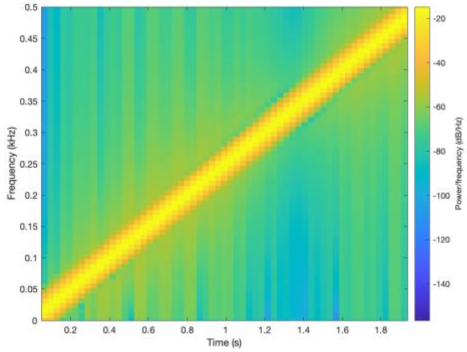
SPFirst Sec. 3-2

(d) $x(t) = \cos(100\pi\, t)\cos(400\pi\, t)$. **Beat frequencies. Signal has sum and difference of 50 Hz and 200 Hz, i.e. 150 Hz and 250 Hz. This is spectrogram (1).**

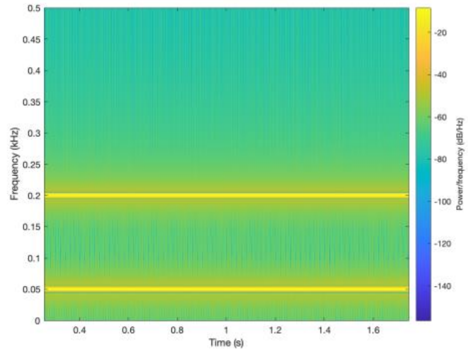
(e) $x(t) = \cos(30e^{2t})$. **Instantaneous frequency is $\frac{d\theta(t)}{dt} = \frac{d}{dt}(30e^{2t}) = 60e^{2t}$ rad/s which is an increasing exponential from 0 Hz to $60e^4$ rad/s (about 521 Hz). This is spectrogram (3).**

(f) **Instantaneous frequency is $\frac{d\theta(t)}{dt} = \frac{d}{dt}(200\pi\, t^2) = 400\pi t$ rad/s which is a line from 0 Hz to 400 Hz for $0 \le t \le 2s$. This is spectrogram (6).**

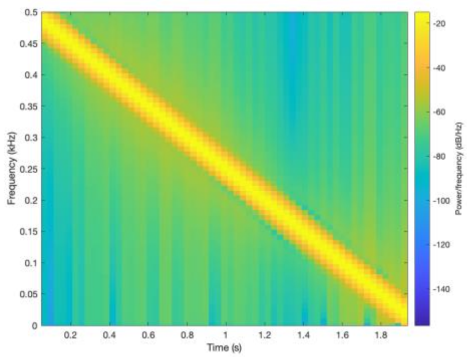
Generating spectrograms for signals in problem 1.4 (Matlab code on next page)



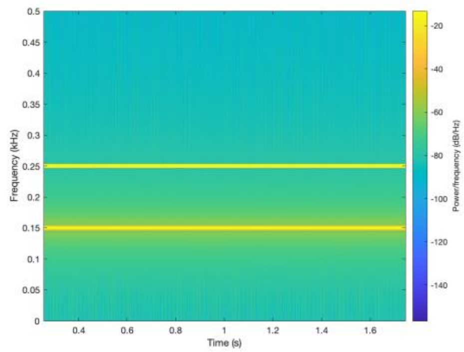
(a) $x(t) = \cos(-250\pi\, t^2)$



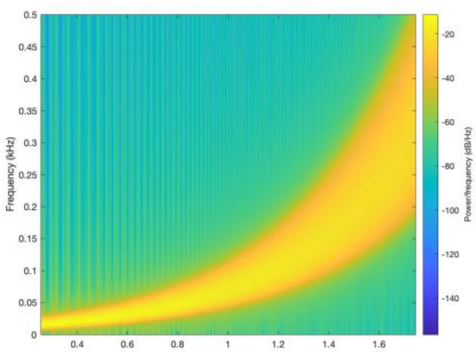
(b) $x(t) = \cos\left(100\pi\, t - \frac{\pi}{4}\right) + \cos(400\pi\, t)$



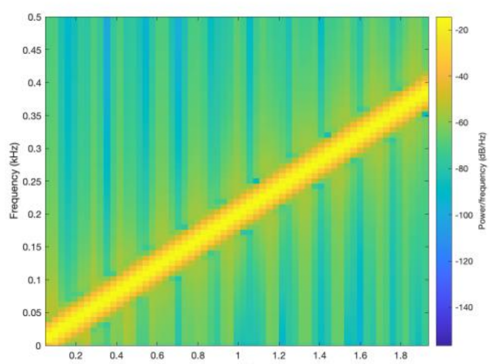
(c) $x(t) = \cos(1000\pi\, t - 250\pi\, t^2)$



(d) $x(t) = \cos(100\pi\, t)\cos(400\pi\, t)$



(e) $x(t) = \cos(30e^{2t})$



(f) $x(t) = \cos(200\pi\, t^2)$

```matlab
%%% Matlab code to generate spectrograms
fs = 2000;
Ts = 1/fs;
t = 0 : Ts : 2;

%%% Spectrogram parameters
blockSize = 1024;
overlap = blockSize - 1;
chirpBlockSize = 256;
chirpOverlap = round(3*chirpBlockSize/4);

%%% (a)
xa = cos(-250*pi*t.^2);
figure;
spectrogram(xa, hamming(chirpBlockSize), chirpOverlap, chirpBlockSize, fs, 'yaxis');
ylim( [0 0.5] );

%%% (b)
xb = cos(100*pi*t - pi/4) + cos(400*pi*t);
figure;
spectrogram(xb, hamming(blockSize), overlap, blockSize, fs, 'yaxis');
ylim( [0 0.5] );

%%% (c)
xc = cos(1000*pi*t - 250*pi*t.^2);
figure;
spectrogram(xc, hamming(chirpBlockSize), chirpOverlap, chirpBlockSize, fs, 'yaxis');
ylim( [0 0.5] );

%%% (d)
xd = cos(100*pi*t) .* cos(400*pi*t);
figure;
spectrogram(xd, hamming(blockSize), overlap, blockSize, fs, 'yaxis');
ylim( [0 0.5] );

%%% (e)
xe = cos(30*exp(2*t));
figure;
spectrogram(xe, hamming(blockSize), overlap, blockSize, fs, 'yaxis');
ylim( [0 0.5] );

%%% (f)
xf = cos(200*pi*t.^2);
figure;
spectrogram(xf, hamming(chirpBlockSize), chirpOverlap, chirpBlockSize, fs, 'yaxis');
ylim( [0 0.5] );
```